S.-T. Yau College Student Mathematics Contests 2025

Analysis and Differential Equations

Individual

Please solve at least three out of the following four problems.

1. Let $\alpha > 0$ be an irrational number. For any $(a, b) \subset [0, 1]$, prove that

$$\lim_{N \to +\infty} \frac{1}{N} \# \big\{ n \in [0, N], n \text{ integer}, \{n\alpha\} \in [a, b] \big\} = b - a$$

where $\{n\alpha\} = n\alpha - \lfloor n\alpha \rfloor$ is the fractional part of $n\alpha$.

2. Let

$$\Omega = \{ \, x + iy \in \mathbb{C} \, : \, x > 0, y > 0 \, \}.$$

Assume that $f : \overline{\Omega} \to \mathbb{C}$ is a bounded continuous function on $\overline{\Omega}$ and holomorphic on Ω such that

$$\forall x \in [0, +\infty], \quad \max\{|f(x)|, \, |f(ix)|\} \leq \begin{cases} 2 & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Show that

$$\forall x + iy \in \Omega, \quad |f(x + iy)| \leq 2^{\frac{2}{\pi} \arctan \frac{1}{2xy}}.$$

3. Suppose that V is a closed subspace of $L^2([0, 1])$ and $V \subset C([0, 1])$. Show that $\dim V < \infty$.

4. Consider a smooth and non-negative function $u(x)$ on $\mathbb{R}^n$ with integer $n \geq 2$. Supposing that $\Delta^2 u = \Delta(\Delta u) = 0$ where $\Delta u = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u$, prove that $u(x)$ is a quadratic polynomial with the form

$$u(x) = \sum_{i=1}^n a_i (x_i - b_i)^2 + a_0$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $a_i \geq 0$, and b_i are some constants.